Diagonalization theorems for matrices over certain domains

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Introduction

In [7] NORDGREN proved a diagonalization theorem for matrices over H^∞, the set of all bounded analytic functions on the unit disc. Making use of this result MOORE and NORDGREN gave, in [6], a new approach to the Jordan model theory of $\mathscr{C}_0$ contractions of finite defect [9—16] and established a conjecture of SZ.-NAGY and FOIAŞ [14]. In the present paper we prove an abstract algebraic generalization of Nordgren's diagonalization theorem.

0. Preliminaries

Let R be a *domain*, i.e. a commutative ring with identity 1 and without zero divisors.[1] Two $m\times n$ matrices A and B over R are said to be *equivalent* if there exist invertible $m\times m$ and $n\times n$ matrices X and Y over R such that $XAY=B$.

We set the following condition:

(*GCD*) In R any two elements have a greatest common divisor (g.c.d.).

It follows from (*GCD*) by induction that any finite system of elements $a_1, \dots, a_n$ from R has a g.c.d. in R. This shall be denoted by $a_1\wedge \dots \wedge a_n$.[2] For any $m\times n$ matrix A over R and any integer k such that $1\leqq k\leqq \min(m, n)$, $\mathscr{D}_k(A)$ will denote the g.c.d. of all minors of order k of A. Set $\mathscr{D}_0(A)=1$. It is easy to see that if $\mathscr{D}_{k-1}(A)=0$ for some k $(k\leqq \min(m, n))$ then $\mathscr{D}_k(A)=0$ as well. For any k such that $1\leqq k\leqq \min(m, n)$ we set $\mathscr{E}_k(A)=\mathscr{D}_k(A)/\mathscr{D}_{k-1}(A)$ with the convention that $\mathscr{E}_k(A)=0$ if $\mathscr{D}_{k-1}(A)=0$.[3] $\mathscr{E}_k(A)$ is called the *invariant factor* of kth order of A. Relying on elementary determinant theory one can easily see that if A and B are two equivalent matrices over R,

[1]) For the algebraic notions we refer the reader to [4].

[2]) $a_1\wedge\dots\wedge a_n$ is determined up to invertible factors.

[3]) The elementary theory of determinants shows that $\mathscr{D}_{k-1}(A)|\mathscr{D}_k(A)$ for $k=1, \dots, \min(m, n)$.

then $\mathscr{E}_k(A)$ equals $\mathscr{E}_k(B)$ up to invertible factors. The matrix A is said to be *normal* if all but possibly the diagonal entries of it vanish and each diagonal entry is a multiple of the preceding one. It is evident that the invariant factor of kth order of a normal matrix equals its diagonal entry in the kth row.

1. An equivalence theorem for matrices over certain domains

There is a classical result [4] which asserts that if R is a principal ideal domain, then every $m \times n$ matrix over R is equivalent to a normal one. Now we are going to prove the analogue of this theorem for domains R having property (GCD) and the following one:

(L) If a and b are relatively prime[4] elements of R (in symbols $a \perp b$) then there exists an element y in R such that $a+by$ is invertible.

At the end of section 3 we shall see (cf. footnote [15]) that our theorem is not a special case of the classical one. On the other hand, the ring of all rational integers does not satisfy (L), so our theorem does not contain the classical one as a special case.

Theorem 1. *If a domain R has properties (GCD) and (L), then any $m \times n$ matrix A over R is equivalent to a normal one.*

Proof. Given any integer j, $1 \leqq j \leqq m$, there exists a matrix A'[5] having the following properties:

(R_j) 1) a'_{j1} divides all entries in the jth row of A';

2) the g.c.d. of all entries in an arbitrary row of A equals the g.c.d. of all entries in the corresponding row of A';

3) A' is equivalent to A.

In fact, from (GCD) and (L) it follows by induction that there exist elements $r_2, \ldots, r_n$ of R such that

$$a_{j1} + r_2 a_{j2} + \ldots + r_n a_{jn} = a_{j1} \wedge \ldots \wedge a_{jn}.$$

Let A' be the matrix obtained from A by adding to its first column the linear combination of its last $n-1$ columns with the coefficients $r_2, \ldots, r_n$. Then the first two requirements in (R_j) are obviously fulfilled. On the other hand, it is an elementary fact that

[4]) I.e. $c \in R$ and $c|a$, $c|b$ imply that c^{-1} exists in R.

[5]) Matrices shall be denoted by Roman capital letters, their entries by the corresponding low case letters, with two subscripts the first one indicating the row.

A' can be obtained from A by multiplying A by a nonsingular matrix from the right, hence 3) in (R_j) also holds.

Now suppose that $j-1 \geqq 1$. Then there exists a matrix A'' having the following properties:

(C_j) 1) $a''_{j-1,1} | a''_{j,1}$;

2) the jth and the subsequent rows are the same in A'' as in A;

3) A'' is equivalent to A.

In fact, (GCD) and (L) assure the existence of an element s of R such that $a_{j-1,1}+sa_{j1}=a_{j-1,1}\wedge a_{j1}$. Let A'' be the matrix obtained from A by adding s times its $(j-1)$th row to its jth row. Then 1) and 2) of (C_j) are obviously satisfied. On the other hand, A'' can be obtained from A by multiplying A by a non-singular matrix from the left, thus 3) in (C_j) also holds true.

Relying on the preceding observations, we are now going to show the existence of a matrix equivalent to the given matrix A and whose entry in the left upper corner divides all its other entries. To this effect, we replace the matrix A by a matrix A' having property (R_m) and denote this matrix A' again by A. Having done this, we replace the new A by a matrix A'' having property (C_m) and denote the replacing matrix again by A. Continuing, we alternately replace the current A by a matrix A' or A'' having successively the properties (R_{m-1}), (C_{m-1}), (R_{m-2}), (C_{m-2}), ..., (R_2), (C_2), (R_1). It is easy to see that the matrix A obtained in the last $\big((2m-1)\text{th}\big)$ step has the property $a_{11} | a_{ik}$ $(1 \leqq i \leqq m,\ 1 \leqq k \leqq n)$.

Subtracting appropriate scalar multiples of the first row (column) of A from the other rows (columns) of A, we can end up with a matrix, denoted by A again, all of whose entries in the first row and column except possibly the one in the left upper corner are zeros. It is an elementary fact that our new A is equivalent to the old one. On the other hand, it is obvious that $a_{11} | a_{ik}$ $(1 \leqq i \leqq m,\ 1 \leqq k \leqq n)$ is still true.

We can accomplish the proof of Theorem 1 in two ways. Either we use induction on m and n or we employ the preceding method $\min(m-1, n-1)$ times more. This part of our proof is routine, so we omit it.

2. General diagonalization theorems

We want to prove a diagonalization theorem for matrices over some domains R having the property (GCD) and a property weaker than (L). For any fixed non-zero element ψ of R the following property is weaker than (L):

$(L\psi)$ If $a, b \in R$ and $a \perp b$, then there are elements x, y in R such that $xa+yb \perp \psi$ and $x \perp \psi$.

Putting $\psi=0$, we obtain a property equivalent to (L). It is obvious that property (L) implies property $(L\psi)$ for any $\psi\in R$. On the other hand, the union of properties $(L\psi)$ $(\psi\neq 0)$ is strictly weaker than property (L). In fact, it can be shown that every principal ideal domain, or more generally, every domain having properties (GCD), (ψ) $(RP\psi)$ and (A) (see section 3) has property $(L\psi)$ for each $\psi\in R$, $\psi\neq 0$, however, for example, the domain of all rational integers does not satisfy property (L).

In the sequel we shall consider some quotient rings of R, $R\neq\{0\}$. Let us fix a non-zero element ψ of our domain R [6]) and suppose that R has property

$(RP\psi)$ For any elements a, b in R, $a\perp\psi$ and $b\perp\psi$ imply $ab\perp\psi$.

Consider the quotient field $\tilde{R}$ of R. Making use of $(RP\psi)$ it can be easily verified that the set R_ψ of all elements t of R that can be written in the form $x=ab^{-1}, a, b\in R$, $b\perp\psi$ is a domain containing R. We can easily see that if R has property $(L\psi)$, then R_ψ has property (L). If R also has property (GCD) then so does R_ψ. In fact, let $x=ab^{-1}$ and $y=cd^{-1}$ $\left(a, b, c, d\in R,\ b, d\overset{R}{\perp}\psi\right)$ [7]) be two elements of R_ψ. We shall show that $x\overset{R_\psi}{\wedge}y$ exists and equals $a\overset{R}{\wedge}c$. From $(L\psi)$ there follows the existence of elements x, y, s of R such that $xa+yc=(a\overset{R}{\wedge}c)s$, $s\overset{R}{\perp}\psi$. Rewriting this as $(xs^{-1})a+(ys^{-1})c=a\overset{R}{\wedge}c$, we can see that if $t\in R_\psi$ is a common divisor of a and c in R_ψ then $t\overset{R_\psi}{\mid}(a\overset{R}{\wedge}c)$. This, together with the obvious relation $(a\overset{R}{\wedge}c)\overset{R_\psi}{\mid}a, c$ means that $a\overset{R_\psi}{\wedge}c$ exists and equals $a\overset{R}{\wedge}c$. Since x (resp. y) differs from a (resp. b) in an invertible factor only, this proves our assertion about $x\overset{R_\psi}{\wedge}y$. [8])

The preceding arguments show that if a domain R has property (GCD) and properties $(RP\psi)$ and $(L\psi)$ for some non-zero element ψ of R, then Theorem 1 can be applied to R_ψ. In particular, for every $m\times n$ matrix A over R, Theorem 1 assures the existence of two non-singular matrices X and Y over R_ψ and that of a normal matrix E over R_ψ such that $XA=EY$. From our reasoning about the g.c.d. in R_ψ and from Preliminaries it follows that E can be chosen to be equal to the diagonal

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[6]) For $\psi=0$ property $(RP\psi)$ always holds and our results in this section are still true but they are equivalent to those of the preceding section.

[7]) $d\overset{R}{\perp}\psi$ means that b is relatively prime to ψ over R. In the sequel, if misunderstanding were possible, we shall indicate by a superscript the domain over which the symbols $\perp$, $\wedge$ or $\mid$ shall be meant.

[8]) I am indebted to my colleague G. Pollák for a remark which enabled me to shorten the proof of the fact that R_ψ has property (GCD). Originally, I derived (GCD) for R_ψ from $(RP\psi)$ and (GCD) only.

matrix of the invariant factors of A over R.[9]) The entries of X and Y are fractions whose numerators and denominatiors are elements of R, the latter ones can be supposed to be relatively prime to ψ. Denote by c the product of the denominators (supposed to be relatively prime to ψ) of all entries in X and Y. Then $(RP\psi)$ implies, by induction, that $c \perp \psi$. Furthermore, since the matrix X is invertible over R_ψ, there is an element r of R_ψ such that $r \det X = 1$. Put $\det X = \frac{x'}{x''}$ and $r = \frac{r'}{r''}$, $x', r', x'', r'' \in R$, $x'', r'' \overset{R}{\perp} \psi$. Then we have $x'r' = x''r''$, so $(RP\psi)$ shows that $x' \overset{R}{\perp} \psi$. Similarly, if $\det Y = \frac{y'}{y''}$, $y', y'' \in R$ and $y'' \overset{R}{\perp} \psi$, then $y' \overset{R}{\perp} \psi$. Put $X' = cX$, $Y' = cY$. Then the entries of X' and Y' are from R. Moreover, $\det X' = \frac{x'}{x''} c^m$, $\det Y' = \frac{y'}{y''} c^n$ so, on account of $(RP\psi)$ the relation $x', y', c \overset{R}{\perp} \psi$ implies that $\det X', \det Y' \overset{R}{\perp} \psi$. Since we also have $X'A = EY'$, our result can be summarized in the following:

Theorem 2. *Suppose that R has property (GCD), and for some non-zero*[10]) *element ψ of R, properties $(RP\psi)$ and $(L\psi)$. Then for every $m \times n$ matrix A over R we can find an $m \times m$ matrix X and an $n \times n$ matrix Y over R such that* $\det X, \det Y \perp \psi$ *and $XA = EY$, where E denotes the diagonal matrix of the invariant factors of A.*

We do not know in general whether the diagonal matrix of the invariant factors of A (even if R has properties (GCD) and $(RP\psi)$, $(L\psi)$ for some $\psi \in R$, $\psi \neq 0$) is normal or not. However, if the assumptions of Theorem 2 hold true for $\psi = \mathscr{E}_i(A)\,(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1}$ (provided that not both $\mathscr{E}_i(A)$ and $\mathscr{E}_{i+1}(A)$ vanish), then we have $\mathscr{E}_i(A) \overset{R}{|} \mathscr{E}_{i+1}(A)$. (Of course, if $\mathscr{E}_i(A) = \mathscr{E}_{i+1}(A) = 0$, then $\mathscr{E}_i(A) \overset{R}{|} \mathscr{E}_{i+1}(A)$ obviously holds.) In fact, the arguments preceding Theorem 2 show that for $\psi = \mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1}$ we have $\mathscr{E}_i(A) \overset{R_\psi}{|} \mathscr{E}_{i+1}(A)$, i.e. we have $\mathscr{E}_{i+1}(A) = \frac{r'}{r''} \mathscr{E}_i(A)$ $(r', r'' \in R,\ r'' \overset{R}{\perp} \psi (= \mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1})$. Since

$$\mathscr{E}_i(A)(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A))^{-1} \overset{R}{\perp} \mathscr{E}_{i+1}(A)(\mathscr{E}_i)A) \wedge \mathscr{E}_{i+1}(A))^{-1}$$

[9]) In detail: On account of Preliminaries E necessarily equals one of the diagonal matrices of the invariant factors, over R_ψ, of A. If E' is one of the diagonal matrices of the invariant factors, over R, of A, then our comments on the g.c.d. in R_ψ show that $e_{ii} = \varphi_i e'_{ii}$ for some invertible elements φ_i of R_ψ and for $i = 1, \ldots, \min(m, n)$. Denoting by T the $n \times n$ diagonal matrix of these φ_i's, T is invertible over R_ψ, $E = E'T$, and $XA = E'TY$.

[10]) For $\psi = 0$ Theorem 2 reduces to Theorem 1.

the equality $r'' \mathscr{E}_{i+1}(A)\left(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A)\right)^{-1}=r' \mathscr{E}_i(A)\left(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A)\right)^{-1}$ shows, by $(RP\psi)$, that $\mathscr{E}_i(A)\left(\mathscr{E}_i(A) \wedge \mathscr{E}_{i+1}(A)\right)^{-1}$ is invertible in R. i.e. $\mathscr{E}_i(A) \mid \mathscr{E}_{i+1}(A)$ [11]).

In order to state our general diagonalization theorem we need the following

Definition. Let A and B be two $m \times n$ matrices over a domain R. We say that A is *quasi-equivalent* to B if for any non-zero element ψ of R there exists an $m \times m$ matrix X and an $n \times n$ matrix Y over R such that $XA=BY$ and $\det X$, $\det Y \perp \psi$.

Suppose that R has property $(\psi)\,(RP\psi)$, i.e. property $(RP\psi)$ for *every* $\psi \in R$. Then the arguments used right before Theorem 2 show that A is quasi-equivalent to B over R if and only if A is equivalent to B over every R_ψ ($\psi \in R$, $\psi \neq 0$). Hence in this case quasi-equivalence is an equivalence relation.

Theorem 2 and the remarks after it imply the following.

Theorem 3. *Suppose that the domain R has properties (GCD), $(\psi)\,(RP\psi)$ and $(\psi \neq 0)\,(L\psi)$ [12]). Then, for matrices over R, quasi-equivalence is an equivalence relation. The diagonal matrix of the invariant factors of any matrix over R is normal and it is quasi-equivalent to the matrix considered.*

3. Examples for Theorems 1—3

For two elements f, g of a domain R we write $f \ll g$ if every non-invertible divisor of f has a non-invertible divisor that divides g. It is easy to see that the relation "$\ll$" is transitive. Let us consider the following property:

(A) For every two elements f and g of R with f non-vanishing there are two elements f_s and f_a in R such that $f_s \perp g$, $f_a \ll g$ and $f=f_s f_a$.

We are now going to give a slightly simpler proof for Lemma 3.1 of [7] in a more general situation.

Lemma. *If R has properties (GCD), (A) and $(\psi)\,(RP\psi)$, then R has property $(\psi \neq 0)\,(L\psi)$.*

Proof. Fix a non-vanishing ψ and consider two relatively prime elements a and b of R. Put $\psi=\psi_s\psi_a$, $\psi_a \ll a$, $\psi_s \perp a$ (property (A)!) and $\delta=a+b\psi_s$. We are going to prove that $\delta \perp \psi$, which will complete the proof of our lemma. For any $\omega \in R, \omega \neq 0$ put $\omega_s=\psi_s \wedge \omega$, $\omega_a=\dfrac{\omega}{\psi_s \wedge \omega}$. We have $\omega=\omega_a\omega_s$, $\omega_s \wedge \psi_s$ and $\omega_a \perp \psi_s\omega_s^{-1}$.

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[11]) Professor B. Sz.-Nagy kindly called our attention to the paper [8] from which it follows that if $\bar{i}$ denotes the least common multiple of $1, \ldots, i$, then $\mathscr{E}_i(A) | \bar{i}\, \mathscr{E}_{i+1}(A)$ for $i=1, \ldots, \min(m, n)$ provided that R has properties (GCD) and $(RP\psi)$ for every $\psi \in R$.

[12]) $(\psi \neq 0)$ is a restricted quantifier and $(\psi \neq 0)(L\psi)$ means that $(L\psi)$ holds true for any $\psi \in R$, $\psi \neq 0$.

Suppose now that $\omega | \psi$. We prove that $\omega_a \ll \psi_a$. In fact, it is obvious that $\omega_a | \psi_a \psi_s \omega_s^{-1}$. For any c such that $c | \omega_a$ and c is not invertible, this divisibity relation, $\omega_a \perp \psi_s \omega_s^{-1}$ and (ψ) $(RP\psi)$ imply that c is not relatively prime to ψ_a, which means that $\omega_a \ll \psi_a$. If ω divides δ too, then from the equality $\delta = a + b\psi_s$ and from $\omega_s | \psi_s$ we deduce that $\omega_s | a$. But $\psi_s \perp a$ by the definition of ψ_s, so $\omega_s \perp a$ (since $\omega_s | \psi_s$) and hence ω_s is invertible. On the other hand, since $\omega_a \ll \psi_a$ and $\psi_a \ll a$, we have $\omega_a \ll a$. This means that if $c | \omega_a$ and c is not invertible, then there exists a non-invertible element d in R such that $d | c$ and $d | a$. Since $\delta = a + b\psi_s$ and ω is supposed to divide δ, we have $d | b\psi_s$. Furthermore, $d | a$ and $a \perp \psi_s$ imply $d \perp \psi_s$, which, together with $d | b\psi_s$ and (ψ) $(RP\psi)$, imply that d is not relatively prime to b. This contradicts the fact that $a \perp b$. Hence c and therefore ω_a are invertible. Since ω_s is also invertible, so is $\omega = \omega_a \omega_s$. This completes the proof of the fact that $\psi \perp \delta$.

Theorem 4. *If the domain R has properties (GCD), (ψ) $(RP\psi)$, and (A), then the conclusions of Theorem 3 are true for R.*

Proof. Our assumptions, by the Lemma and Theorem 3, immediately imply the conclusions of Theorem 3.

Let R now be a domain such that every non-zero element of R has a prime factorization. It is obvious that these prime factorizations are, up to invertible factors, uniquely determined by the elements factored and R has properties (GCD) and (ψ) $(RP\psi)$. Moreover, in R the relation $f \ll g$ is equivalent to the following: Every prime divisor of f is a divisor of g, too. For any two $f, g \in R$, $f \neq 0$ define f_s as the product of those prime factors of f (with multiplicity) which do not devide g. Putting $f_a = ff_s^{-1}$ we have $f = f_s f_a$ and $f_s \perp g$, $f_a \ll g$. This shows that R has property (A), too, and we have

Theorem 5. *If in the domain R every non-zero element has a prime factorization, then the conclusions of Theorem 3 are true for R.*

Remark. In this special case our lemma, that is property $(\psi \neq 0)$ $(L\psi)$, could be proved more easily.

Since in any principal ideal domain every non-zero element has a prime factorization, we can consider this result a generalization of a weaker version of the classical theorem which asserts that every matrix over a principal ideal domain is equivalent to a normal one. Let us remark that there are numerous examples of domains which have the prime factorization property and which are not principal ideal domains [1].[13])

[13]) For example, the domain of all polynomials of n variables ($n \geqq 2$) over an arbitrary field, or more generally over a domain which has the prime factorization property is not a principal ideal domain and has the prime factorization property [1].

An appropriate quotient domain of the domain H^∞ of all bounded analytic functions on the unit disc[14]) (from the study of which all our investigations have started) provides an example for a domain R having properties (GCD), (A) and (ψ)($RP\psi$) and not having the prime factorization property. In fact, using the properties of inner functions, it is easy to see that the domain $R=N^+$ of all analytic functions f on the unit disc, of the form $f=g/h$, where $g\in H^\infty$, $h\in H^\infty$ and h is outer, has properties (GCD), (A) and (ψ) ($RP\psi$), and N^+ does not have the prime factorization property[15]).

Let us now compare our results applied to the special case of the domain N^+ with those obtained by Nordgren [7]. He states a theorem about H^∞ and not about the domain N^+ but his notion of divisibility in H^∞ coincides with the usual algebraic notion of divisibility over N^+. If in our Theorem 2 applied to the domain N^+ we multiply all entries of X and Y by the product of the denominators (which can be supposed to be outer functions) of all entries of X and Y, then we obtain Theorem 3.1 of [7]. Moreover, for matrices over H^∞, of equal size $m\times n$, one can introduce the following notion of quasi-equivalence: A is said to be quasi-equivalent to B if for any $\psi\in H^\infty$, $\psi\neq 0$ there exist square matrices X and Y over H such that $XA=BY$ and that $\det X$ and $\det Y$ are relatively prime to ψ over N^+. Theorem 3.1 of [7] asserts then that every matrix over H^∞ is quasi-equivalent to the matrix of its invariant factors. (See also [4].)

By the way, Theorems 2.1 and 3.1 of [7] together with the fact that quasi-equivalence in the above sense is an equivalence relation show that quasi-equivalence as defined in the present paper is the same as that defined in [7].

4. Additional remarks

After I had finished the studies contained in sections 1—3, J. Erdős called my attention to the paper [5] of I. Kaplansky. There it is proved among others a necessary and sufficient condition for a domain to be an *elementary divisor ring* (cf. Theorem 5.2), that is, a ring over which every $m\times n$ matrix is equivalent to a normal one. (Kaplansky studies non-abelian rings, too, in which case the definition of normality has to be modified.) It is easy to check that Kaplansky's conditions are satisfied in any domain having properties (GCD) and (L). So we could shorten the proof of Theorem 1 by a reference to Kaplansky's result. One of Kaplansky's conditions is that in the domain under consideration every finitely generated ideal is principal. It is easy to see that in the domain of all complex polynomials of two variables x and y the ideal generated by x^2, xy and y^2 is not principal, so we cannot generally have equivalence in Theorem 5 and so in Theorems 3—4, either.

[14]) For the notions about H^∞ we refer the reader to [3].

[15]) Neither does any of the domains R_ψ.

Finally, we mention that O. HELMER in his paper [2] introduces the notion of adequate rings. A ring is called an *adequate ring* if it is a domain, every finitely generated ideal in it is principal, and satisfies condition (A) of our paper. Helmer proves that every adequate ring is an elementary divisor ring. We have seen that the domain considered at the end of section 3 has property (A).

Problem. Is the domain N^+ considered at the end of section 3 adequate?

If it were, we would have equivalence in Nordgren's theorem on the domain N^+, instead of quasi-equivalence.

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